Cosmic No Hair for Collapsing Universes

James E. Lidsey

Astronomy Unit, School of Mathematical Sciences
Queen Mary, University of London
Mile End Road, London E1 4NS
United Kingdom

E-mail: `J.E.Lidsey@qmul.ac.uk`

Abstract. It is shown that all contracting, spatially homogeneous, orthogonal Bianchi cosmologies that are sourced by an ultra-stiff fluid with an arbitrary and, in general, varying equation of state asymptote to the spatially flat and isotropic universe in the neighbourhood of the big crunch singularity. This result is employed to investigate the asymptotic dynamics of a collapsing Bianchi type IX universe sourced by a scalar field rolling down a steep, negative exponential potential. A toroidally compactified version of M*-theory that leads to such a potential is discussed and it is shown that the isotropic attractor solution for a collapsing Bianchi type IX universe is supersymmetric when interpreted in an eleven-dimensional context.

1. Introduction

The possibility that our universe underwent a 'pre-big bang' contraction before bouncing into its present expansionary phase continues to attract attention (see, e.g., [1, 2, 3]). In scenarios of this type, a central question to address is the behaviour of the universe during the final stages of the collapse in the vicinity of the big crunch singularity. Recently, Erickson *et al.* [4] have considered the class of collapsing, spatially homogeneous cosmologies that are sourced by an ultra-stiff perfect fluid with an equation of state $p = (\gamma - 1)\rho$, where $\gamma > 2$. In the case of a constant equation of state parameter, they have proved that the universe asymptotes to the spatially flat and isotropic Friedmann-Robertson-Walker (FRW) cosmology on the approach to the big crunch. This is to be expected on qualitative grounds, since the energy density of the matter source scales as $\rho \propto a^{-3\gamma}$, whereas the curvature and anisotropies vary as a^{-2} and a^{-6}, respectively. The latter therefore become subdominant as the spatial volume $a^3 \to 0$. More generally, for a variable equation of state, they have argued that if the spatial curvature and anisotropy are initially small, they remain so during the collapse.

In this paper we provide a rigorous proof of a 'cosmic no hair' result for all contracting, orthogonal, spatially homogeneous Bianchi cosmologies, where the matter source has an arbitrary equation of state $\gamma = \gamma(\rho)$ that is subject only to the condition that it is differentiable and bounded such that $(\gamma - 2)$ is positive-definite for all time (including at the big crunch). One example of a matter source with an equation of state $\gamma > 2$ is a minimally coupled scalar field with a negative-definite self-interaction potential. We employ the cosmic no hair theorem to gain insight into the behaviour of a collapsing Bianchi type IX cosmology sourced by a scalar field interacting through a steep, negative exponential potential. We find that such a universe can indeed isotropize at the big crunch.

Negative exponential potentials are known to arise in compactifications of higher-dimensional theories such as M*-theory [5]. This is an eleven-dimensional theory in a spacetime with signature (9+2) and is directly related to M-theory [6] by string dualities. For a particular toroidal compactification of this theory to four dimensions, we interpret the isotropic attractor solution for a contracting Bianchi type IX cosmology in an eleven-dimensional context and find that it corresponds to a supersymmetric background.

2. Cosmic No hair Theorem

2.1. Bianchi Cosmology

Bianchi models are spatially homogeneous cosmologies admitting a three–parameter local group G_3 of isometries that acts simply transitively on spacelike hypersurfaces Σ_t. Coordinates can be chosen such that the four–dimensional line element has the form $ds^2 = -dt^2 + h_{ab}(t)\omega^a\omega^b$ $(a, b = 1, 2, 3)$, where the one–forms ω^a satisfy the Maurer–Cartan equation $d\omega^a = \frac{1}{2}C^a{}_{bc}\omega^b \wedge \omega^c$ and $C^a{}_{bc}$ are the structure constants of the Lie algebra of G_3. Since $C^a{}_{(bc)} = 0$, $C^a{}_{bc}$ has at most nine independent components and

these are classified in terms of a symmetric 3×3 matrix, n^{ab}, and the components of a 3×1 vector $A_b \equiv C^a{}_{ab}$. This implies that the structure constants can be expressed in the form

$$C^c{}_{ab} \equiv n^{cd}\varepsilon_{dab} + \delta^c_{[a}A_{b]}, \tag{1}$$

where ε^{abc} is the totally antisymmetric tensor with $\varepsilon^{123} = 1$. Substitution of Eq. (1) into the Jacobi identity $C^e{}_{d[a}C^d{}_{bc]} = 0$ then implies that A_b is transverse to n^{ab}:

$$n^{ab}A_b = 0. \tag{2}$$

If $A_b \neq 0$, it represents an eigenvector of n^{ab} with zero eigenvalue and it may be assumed without loss of generality that $n^{ab} = \mathrm{diag}[n_1, n_2, n_3]$ and that $A_b = (A, 0, 0)$. Moreover, a suitable rescaling can always be found such that the eigenvalues of n^{ab} take values $\{0, \pm 1\}$. As a result, Eq. (2) simplifies to

$$n_1 A = 0. \tag{3}$$

Bianchi class A models satisfy $A = 0$ and the class B are defined by the property $A \neq 0$ $(n_1 = 0)$ [7].

Throughout this paper, we consider orthogonal Bianchi models where the fluid velocity vector is orthogonal to the group orbits. In the case of a scalar field matter source, this implies that the field is constant on the surfaces of homogeneity. We employ the orthonormal frame approach developed by Wainwright and collaborators [8, 10]. (For a review see [11]). In this approach, the Hubble scalar is defined by $H \equiv \dot{\ell}/\ell$, where ℓ is a length scale and a dot denotes differentiation with respect to t. The Raychaudhuri and Friedmann equations are then given by

$$\dot{H} = -H^2 - \frac{2}{3}\sigma^2 - \frac{1}{6}(3\gamma - 2)\rho \tag{4}$$

$$3H^2 = \sigma^2 - \frac{1}{2}{}^{(3)}R + \rho, \tag{5}$$

respectively, where $\sigma^2 \equiv \frac{1}{2}\sigma_{ab}\sigma^{ab}$ is defined in terms of the shear tensor σ_{ab}, ${}^{(3)}R$ is the scalar curvature of the t = constant hypersurfaces, and ρ and p represent the energy density and pressure of the fluid. The Friedmann constraint equation (5) may be expressed in the form

$$\Omega + \Sigma^2 + K = 1 \tag{6}$$

by defining expansion-normalized density, shear and curvature parameters:

$$\Omega \equiv \frac{\rho}{3H^2}, \quad \Sigma^2 \equiv \frac{\sigma^2}{3H^2}, \quad K = -\frac{{}^{(3)}R}{6H^2}. \tag{7}$$

A deceleration parameter, $q \equiv -\ddot{\ell}\ell/\dot{\ell}^2$, may also be defined such that

$$\dot{H} = -(1+q)H^2 \tag{8}$$

and Eqs. (4) and (8) together imply that

$$q = 2\Sigma^2 + \frac{1}{2}(3\gamma - 2)\Omega. \tag{9}$$

Finally, the covariant conservation of energy-momentum is expressed in the form of the fluid equation:

$$\dot{\rho} = -3H\gamma\rho, \tag{10}$$

where $p \equiv [\gamma(\rho) - 1]\rho$ defines the equation of state parameter, $\gamma(\rho)$. We will consider an arbitrary equation of state subject only to the conditions that $p = p(\rho)$ is at least $\mathcal{C}^1$ and that $(\gamma - 2)$ is positive-definite for all values of the spatial volume. We also assume the standard energy conditions hold and, in particular, that $\rho \geq 0$.

The scalar curvature satisfies ${}^{(3)}R \leq 0$ for all Bianchi types except the type IX. It then follows from the Friedmann equation (5) that the spatial volume of an initially contracting Bianchi type I-VIII universe will decrease monotonically with cosmic time, t. In view of this, it proves convenient to define a dimensionless time variable τ [8, 10]:

$$\frac{dt}{d\tau} \equiv \frac{1}{H}. \tag{11}$$

Hence, τ is a monotonically decreasing function of t when $H < 0$ and, since $0 < \ell < \infty$, the big crunch singularity will occur at $\tau \to -\infty$.

We now show that $\Omega \to 1$ in the vicinity of the big crunch for all Bianchi type I-VIII universes. Firstly, Eq. (6) implies that $\Omega \leq 1$ since $K \geq 0$. Furthermore, Eqs. (6), (7), (8) and (10) yield an evolution equation for the density parameter which takes the form:

$$\Omega' = \left[-(3\gamma - 2)K + 3(2-\gamma)\Sigma^2\right]\Omega, \tag{12}$$

where a prime denotes $d/d\tau$. It follows from Eq. (12) that $\Omega' \leq 0$ for any initially contracting Bianchi type I-VIII universe, with equality iff $K = \Sigma^2 = 0$ for any non–vacuum orbit ($\Omega > 0$). Hence, Ω is a monotonic decreasing function of τ and, since Ω is bounded, we may conclude that $\lim_{\tau\to-\infty}\Omega' = 0$. Eq. (12) then implies that

$$\lim_{\tau\to-\infty} K = \lim_{\tau\to-\infty} \Sigma = 0, \qquad \lim_{\tau\to-\infty} \Omega = 1, \tag{13}$$

where the latter limit follows directly from Eq. (6).

To proceed further, we will require the specific form of the Einstein field equations for each Bianchi model. For the Bianchi type I-VIII universes, these equations can be expressed in the form of an autonomous set of ordinary differential equations (ODEs) of the form $\mathbf{x}' = \mathbf{f}(\mathbf{x})$, subject to a constraint equation $g(\mathbf{x}) = 0$, where the state vector $\mathbf{x} \in \Re^6$. The physical interpretation of the state variables $\mathbf{x}$ is different for the class A and B models, however, and we therefore consider each class in turn in the following Subsections.

2.2. Bianchi Class A (I - VIII)

The physical state of a Bianchi class A cosmology is determined by the vector $\mathbf{x} = (H, \Sigma_+, \Sigma_-, N_1, N_2, N_3)$, where $N_a \equiv n_a/H$, $\Sigma_\pm = \sigma_\pm/H$ and $\sigma_\pm$ are linear combinations of the two independent components of the (traceless) shear tensor [9, 8]. Thus, $\Sigma_\pm$

determine the anisotropy associated with the Hubble flow and are related to the shear parameter by

$$\Sigma^2 = \Sigma_+^2 + \Sigma_-^2, \tag{14}$$

whereas N_a parametrize the spatial curvature of the group orbits and are related to the curvature parameter such that

$$K = \frac{1}{12}\left[N_1^2 + N_2^2 + N_3^2 - 2(N_1N_2 + N_2N_3 + N_3N_1)\right]. \tag{15}$$

The Einstein field equations for the Bianchi class A are then given by an autonomous set of first-order ODEs [8]:

$$\Sigma'_\pm = -(2-q)\Sigma_\pm - S_\pm \tag{16}$$
$$N'_1 = (q - 4\Sigma_+)N_1 \tag{17}$$
$$N'_2 = (q + 2\Sigma_+ + 2\sqrt{3}\Sigma_-)N_2 \tag{18}$$
$$N'_3 = (q + 2\Sigma_+ - 2\sqrt{3}\Sigma_-)N_3, \tag{19}$$

where

$$S_+ = \frac{1}{6}\left[(N_2 - N_3)^2 - N_1(2N_1 - N_2 - N_3)\right] \tag{20}$$
$$S_- = \frac{1}{2\sqrt{3}}(N_3 - N_2)(N_1 - N_2 - N_3), \tag{21}$$

together with the decoupled equation

$$H' = -(1+q)H. \tag{22}$$

Eqs. (13) and (14) imply immediately that $\lim_{\tau\to-\infty}\Sigma_+ = \lim_{\tau\to-\infty}\Sigma_- = 0$ and it follows from Eq. (9) that $\lim_{\tau\to-\infty} q = (3\gamma-2)/2 > 0$. By following a similar argument to that of [12] (who consider expanding cosmologies with $0 \le \gamma < 2/3$), we may then deduce from Eqs. (17)–(19) that for each N_a there exists a parameter $\varepsilon > 0$ such that $N'_a/N_a > \varepsilon$ for a sufficiently negative τ, and hence that $\lim_{\tau\to-\infty} N_a = 0$. We conclude, therefore, that the spatially flat and isotropic FRW universe is the global sink for all ultra-stiff, orthogonal, initially contracting Bianchi models of type I-VIII (class A).

2.3. Bianchi Class B

For the class B models, we employ the framework developed by Hewitt and Wainwright [10] and define the quantities $\tilde{\sigma} \equiv \frac{1}{6}\tilde{\sigma}_{ab}\tilde{\sigma}^{ab}$ and $\sigma_+ \equiv \frac{1}{2}\sigma^a{}_a$, where $\tilde{\sigma}_{ab}$ is the trace-free part of the shear tensor σ_{ab}. Likewise, we define $\tilde{n} \equiv \frac{1}{6}\tilde{n}_{ab}\tilde{n}^{ab}$ and $n_+ \equiv \frac{1}{2}n^a{}_a$, where $\tilde{n}_{ab}$ is the trace-free part of n_{ab}. Since $A \neq 0$ for this class, there exists a constant $\tilde{h}$ such that $\tilde{n} = \frac{1}{3}(n_+^2 - \tilde{h}A^2)$. For $n_2 n_3 \neq 0$, this defines the group parameter $h = \tilde{h}^{-1}$ for types VI$_h$ ($h < 0$) and VII$_h$ ($h > 0$), respectively. (Types IV and V correspond to $\tilde{h} = 0$ and the Bianchi type III is the same as the type VI$_{-1}$).

The specific form of the Einstein field equations is unimportant in establishing the nature of the attractor solution for ultra-stiff, collapsing Bianchi class B universes. Indeed, it is sufficient to define Hubble normalized variables [9, 10] such that $\Sigma_+ =$

σ_+/H, $\tilde{\Sigma} = \tilde{\sigma}/H$, $\tilde{A} = A^2/H^2$, $N_+ = n_+/H$ and $\tilde{N} = \frac{1}{3}(N_+^2 - \tilde{h}\tilde{A})$, where $\tilde{A} \geq 0$, $\tilde{\Sigma} \geq 0$, and $\tilde{N} \geq 0$. In this case, the evolution equation for the density parameter, Ω, still has the form given by Eq. (12), where the deceleration parameter, q, is defined by Eq. (9), but the variables $\{K, \Sigma^2\}$ in these expressions are now defined by

$$K = \tilde{N} + \tilde{A} \tag{23}$$

$$\Sigma^2 = \Sigma_+^2 + \tilde{\Sigma}. \tag{24}$$

Inspection of Eq. (13) therefore implies that as the universe approaches the big crunch, $\lim_{\tau\to-\infty} \Sigma_+ = \lim_{\tau\to-\infty} \tilde{\Sigma} = 0$ and, since $\tilde{N}$ and $\tilde{A}$ are both non-negative quantities, it also follows that $\lim_{\tau\to-\infty} \tilde{N} = \lim_{\tau\to-\infty} \tilde{A} = 0$. Moreover, the dimensionless curvature variables for the Bianchi class B are given by

$$\mathcal{S}_+ = 2\tilde{N}, \qquad \tilde{\mathcal{S}}^{ab}\tilde{\mathcal{S}}_{ab} = 24(\tilde{A} + N_+^2)\tilde{N}, \tag{25}$$

from which we deduce that $\lim_{\tau\to-\infty} \mathcal{S}_+ = \lim_{\tau\to-\infty} \tilde{\mathcal{S}}^{ab}\tilde{\mathcal{S}}_{ab} = 0$.

In the Bianchi class B, there is a special model corresponding to the type $\mathrm{VI}_{-1/9}$, which is exceptional in the sense that its isometry group does not necessarily admit an Abelian subgroup that acts orthogonally transitively. Nonetheless, Eqs. (12) and (13) and Eqs. (23)-(25) remain valid for the exceptional type $\mathrm{VI}_{-1/9}$ and the above analysis therefore applies to all Bianchi class B models. (More precisely, the shear parameter (24) acquires additional terms due to the extra independent component, but it may still be expressed as a sum of non-negative terms that must all vanish when Eq. (13) holds [13]). We may conclude, therefore, that the spatially flat and isotropic FRW universe is the global sink for all ultra-stiff, orthogonal, initially contracting Bianchi class B cosmologies.

2.4. Bianchi Type IX

It now only remains to consider the Bianchi type IX universe. The physical state of a Bianchi type IX model is parametrized by the state vector $\mathbf{x} = (H, \sigma_+, \sigma_-, n_1, n_2, n_3) \in \Re^6$ as for other types in the class A. However, since $n_1 > 0$, $n_2 > 0$, and $n_3 > 0$ for this model, the curvature parameter K is no longer semi-positive definite and, consequently, the Hubble parameter may not necessarily be a monotonically varying function of time. This implies that alternatives to the Hubble-normalized variables are required for a global analysis. A set of appropriate variables that compactifies the Bianchi type IX phase space was introduced by Hewitt, Uggla and Wainwright (see section 8.5.2 of [11]). In this approach, physical quantities are normalized in terms of a function, D, such that

$$(\bar{H}, \bar{\Sigma}_\pm, \bar{N}_a, \bar{\Omega}) \equiv \left(\frac{H}{D}, \frac{\sigma_\pm}{D}, \frac{n_a}{D}, \frac{\rho}{3D^2}\right), \tag{26}$$

where

$$D \equiv \sqrt{H^2 + \frac{1}{4}(n_1 n_2 n_3)^{2/3}}. \tag{27}$$

In particular, Eq. (27) implies that $\bar{H}$ is bounded by the constraint equation

$$\bar{H}^2 + \frac{1}{4}(\bar{N}_1\bar{N}_2\bar{N}_3)^{2/3} = 1. \tag{28}$$

A new time variable, $\bar{\tau}$, is also defined:

$$\frac{dt}{d\bar{\tau}} = \frac{1}{D} \tag{29}$$

and Eqs. (17)-(19), (22) and (27) then imply that the evolution equation for D takes the form

$$D^* = -(1+\bar{q})\bar{H}D, \tag{30}$$

where a star denotes differentiation with respect to $\bar{\tau}$ and $\bar{q} \equiv \bar{H}^2 q$. It can be further shown that the Einstein field equations take the form [11]

$$\bar{H}^* = -(1-\bar{H}^2)\bar{q} \tag{31}$$

$$\bar{\Sigma}^*_\pm = -(2-\bar{q})\bar{H}\bar{\Sigma}_\pm - \bar{S}_\pm \tag{32}$$

$$\bar{N}^*_1 = (\bar{H}\bar{q} - 4\bar{\Sigma}_+)\bar{N}_1 \tag{33}$$

$$\bar{N}^*_2 = (\bar{H}\bar{q} + 2\bar{\Sigma}_+ + 2\sqrt{3}\bar{\Sigma}_-)\bar{N}_2 \tag{34}$$

$$\bar{N}^*_3 = (\bar{H}\bar{q} + 2\bar{\Sigma}_+ - 2\sqrt{3}\bar{\Sigma}_-)\bar{N}_3, \tag{35}$$

where $\bar{S}_\pm$ are defined by Eqs. (20) and (21), respectively, with N_a replaced by $\bar{N}_a$.

Moreover, it follows from Eqs. (9) and (28) that

$$\bar{q} = \frac{1}{2}(3\gamma-2)(1-\bar{V}) + \frac{3}{2}(2-\gamma)\bar{\Sigma}^2, \tag{36}$$

where

$$\bar{\Sigma}^2 = \bar{\Sigma}^2_+ + \bar{\Sigma}^2_- \tag{37}$$

and

$$\bar{V} \equiv \frac{1}{12}\left[\bar{N}^2_1 + \bar{N}^2_2 + \bar{N}^2_3 - 2\bar{N}_1\bar{N}_2 - 2\bar{N}_2\bar{N}_3 - 2\bar{N}_1\bar{N}_3 \right.$$
$$\left. +3(\bar{N}_1\bar{N}_2\bar{N}_3)^{2/3}\right]. \tag{38}$$

The definition (38) implies that $\bar{V} \geq 0$ and substitution of Eq. (15) yields $\bar{V} = \bar{H}^2[K + (N_1N_2N_3)^{2/3}/4]$. Thus, the Friedmann constraint equation (6) may be expressed in the form:

$$\bar{\Sigma}^2 + \bar{V} + \bar{\Omega} = 1, \tag{39}$$

whereas substitution of Eq. (39) into Eq. (36) implies that

$$\bar{q} = 2\bar{\Sigma}^2 + \frac{1}{2}(3\gamma-2)\bar{\Omega}. \tag{40}$$

Finally, we may derive an evolution equation for the density parameter, $\bar{\Omega} = \rho/(3D^2)$, by differentiating with respect to $\bar{\tau}$, and substituting in the fluid equation (10) and the evolution equation (30). We find that

$$\bar{\Omega}^* = \bar{\Omega}\bar{H}\left[-(3\gamma-2)\bar{V} + 3(2-\gamma)\bar{\Sigma}^2\right]. \tag{41}$$

We may now deduce from Eq. (40) that when $(\gamma - 2)$ is positive–definite, $\bar{q} \geq 0$ with equality iff $\bar{\Omega} = \bar{\Sigma}^2 = 0$. Furthermore, since Eq. (28) implies that $\bar{H}$ is bounded, $-1 \leq \bar{H} \leq 1$, it follows from Eq. (31) that $\bar{H}^* \leq 0$ and, consequently, that $\bar{H}$ is

a monotone decreasing function. On the other hand, $\bar{q} > 0$ for a non–vacuum orbit ($\bar{\Omega} > 0$), and this implies that $\lim_{\bar{\tau}\to-\infty} \bar{H} = 1$ and $\lim_{\bar{\tau}\to+\infty} \bar{H} = -1$. In other words, an initially expanding model will eventually undergo a recollapse (when $\bar{H}$ passes through zero). Once the recollapse occurs, Eq. (41) implies that $\bar{\Omega}^* \geq 0$, with equality iff $\bar{V} = \bar{\Sigma}^2 = 0$. However, since $\bar{\Omega} \leq 1$ is bounded due to Eq. (39), we deduce that $\lim_{\bar{\tau}\to+\infty} \bar{\Omega}^* = 0$ and, hence, that

$$\lim_{\bar{\tau}\to+\infty} \bar{\Omega} = 1, \quad \lim_{\bar{\tau}\to+\infty} \bar{V} = 0, \quad \lim_{\bar{\tau}\to+\infty} \bar{\Sigma}^2 = 0. \tag{42}$$

Thus, Eq. (40) implies that $\lim_{\bar{\tau}\to+\infty} \bar{q} = (3\gamma - 2)/2$, from which it follows via Eqs. (33)–(35) that for a sufficiently large $\bar{\tau}$, there exists an $\varepsilon > 0$ such that $d \ln \bar{N}_a / d\bar{\tau} < -\varepsilon$ and therefore that $\lim_{\bar{\tau}\to+\infty} \bar{N}_a = 0$. Consequently, a Bianchi type IX cosmology sourced by an ultra-stiff fluid isotropizes in the same way as the other Bianchi types as it approaches the big crunch at $\bar{\tau} \to \infty$. In particular, this implies that there is no chaotic (oscillatory) behaviour in the vicinity of the singularity.

Since we have now covered all possible Bianchi types, we may summarize the above analysis in the form of a cosmic no hair theorem: *all initially contracting, spatially homogeneous, orthogonal Bianchi type I-VIII cosmologies and all Bianchi type IX universes that are sourced by an ultra-stiff fluid with an equation of state such that* $(\gamma - 2)$ *is positive-definite collapse into an isotropic singularity, where the sink is the spatially flat and isotropic FRW universe.* For the class of models where the equation of state asymptotes to a constant value $\gamma > 2$ on the approach to the singularity, the FRW cosmology is a self-similar power-law solution, where the scale factor varies as $a \propto (-t)^{2/3\gamma}$.

In the following Section, we employ this theorem to determine the nature of a collapsing Bianchi type IX universe with a matter source consisting of a minimally coupled scalar field self-interacting through a negative-definite exponential potential.

3. Isotropization of Collapsing Bianchi type IX Scalar Field Cosmology

In this Section, we consider an action of the form

$$S = \int d^4x \sqrt{-g} \left[R - (\nabla\varphi)^2 - V(\varphi) \right], \qquad V = V_0 \exp(-\lambda\varphi), \tag{43}$$

where V represents the potential of the scalar field, φ, and $V_0 < 0$ and $\lambda > 0$ are constants. We assume the field is orthogonal, i.e., that it is constant on the surfaces of homogeneity, $\varphi = \varphi(t)$, so that its energy density is given by $\rho = \frac{1}{2}\dot{\varphi}^2 + V$. The effective equation of state of the field is defined by

$$\gamma = \frac{2\dot{\varphi}^2}{\dot{\varphi}^2 + 2V} \tag{44}$$

and is therefore bounded such that $\gamma \geq 2$.

The presence of a negative potential energy implies that not all the terms on the right-hand side of the Friedmann equation (5) will be semi-positive definite. Thus, a given Bianchi model may undergo a recollapse and, consequently, the Hubble normalized

variables do not lead to a global compact phase space. Nonetheless, the theorem developed in the previous Section can be employed to determine the nature of collapsing, homogeneous scalar field cosmologies in the vicinity of the big crunch singularity.

It is known that the collapsing Bianchi I model is stable to curvature and anisotropy perturbations near the singularity [14]. In this Section, we will consider the Bianchi type IX universe. In this case, variables parametrizing the evolution of the scalar field can be defined such that [15]

$$\bar{\Psi} \equiv \frac{\dot{\varphi}}{\sqrt{6}D}, \qquad \bar{\Theta} \equiv \frac{\sqrt{-V}}{\sqrt{3}D}. \tag{45}$$

The evolution equations for these variables are then determined by the scalar field equation, $\dot{\rho} = -3H\dot{\phi}^2$, which itself follows as a consequence of energy-momentum conservation. We find that

$$\bar{\Psi}^* = (\bar{q} - 2)\bar{H}\bar{\Psi} - \frac{\sqrt{6}\lambda}{2}\bar{\Theta}^2 \tag{46}$$

$$\bar{\Theta}^* = \left[(1 + \bar{q})\bar{H} - \frac{\sqrt{6}\lambda}{2}\bar{\Psi}\right]\bar{\Theta}. \tag{47}$$

It also proves convenient to define a new bounded variable [15]:

$$d \equiv \frac{D}{D+1}, \tag{48}$$

where $0 \leq d \leq 1$. This evolves such that

$$d^* = (1 + \bar{q})\bar{H}d(d - 1) \tag{49}$$

and the big crunch singularity, $\bar{H} \to -1$, corresponds to $D \to \infty$ ($d \to 1$).

The physical state of a Bianchi type IX scalar field cosmology is therefore given by the vector $\mathbf{x} = (d, \bar{H}, \bar{\Sigma}_+, \bar{\Sigma}_-, \bar{N}_1, \bar{N}_2, \bar{N}_3, \bar{\Psi}, \bar{\Theta})$ and the corresponding autonomous set of ODEs are Eqs. (31)-(35) and Eqs. (46), (47) and (49), subject to the constraint equations (28) and (39). The density parameter is $\bar{\Omega} = \bar{\Psi}^2 - \bar{\Theta}^2$ and evolves according to Eq. (41).

We will assume that the universe is undergoing a collapse, $\bar{H} < 0$, and that the field has positive energy density, $\bar{\Omega} > 0$. It is anticipated that this should represent the behaviour of generic solutions once the recollapse has set in. Since $\bar{q} > 0$, Eq. (31) implies that $\bar{H}^* < 0$ and, consequently, Eq. (41) implies that $\bar{\Omega}$ continues to grow monotonically. Thus, the energy density of the field remains positive, $\dot{\phi}^2 > 2|V|$, and the sign of $\bar{\Psi}$ is fixed on the approach to the big crunch. Moreover, since $\bar{\Omega} > 0$ and $\bar{q} > 0$, the variables $\{\bar{\Sigma}^2_\pm, \bar{V}, \bar{\Omega}\} \leq 1$ (due to Eq. (39)) and $\lim_{\bar{\tau}\to\infty} \bar{H} = -1$.

The cosmic no hair theorem of Section II will apply for this collapsing Bianchi type IX scalar field cosmology if it can be shown that $\gamma > 2$ at the singularity. In principle, however, a scalar field may become dominated by its kinetic energy during the collapse, in which case $\gamma \to 2$ and $\bar{\Theta} \to 0$. This implies that the shear may not necessarily become sub-dominant. In order to establish whether this possibility arises, we will first show that if $\lim_{\bar{\tau}\to\infty} \gamma = 2$, the contracting Bianchi IX model asymptotes in the vicinity of the big crunch toward the spatially flat type I background, where $\bar{N}_1 = \bar{N}_2 = \bar{N}_3 = 0$.

Firstly, an argument similar to that presented after Eq. (41) implies that $\lim_{\bar{\tau}\to\infty} \bar{V} = 0$. However, for an equilibrium point with $\gamma = 2$, Eq. (36) implies that $\bar{q} = 2$. It then follows from Eq. (32) that this equilibrium point will also have $\bar{S}_{\pm} = 0$. The definition (21) then requires that $\bar{N}_3 = \bar{N}_2$ or $\bar{N}_1 = \bar{N}_2 + \bar{N}_3$. If the latter condition applies, the requirements that $\bar{V} = \bar{S}_+ = 0$ then imply that $\bar{N}_2 = 0$ (assuming without loss of generality that $\bar{N}_3 > \bar{N}_2 > \bar{N}_1$). On the other hand, if $\bar{N}_1 = \bar{N}_3 \neq 0$, Eqs. (33) and (35) imply that $\bar{\Sigma}_+ = -1/2$ and $\bar{\Sigma}_- = -\sqrt{3}/2$. In other words, the equilibrium point would correspond to a vacuum model, where $\bar{\Psi} = \bar{\Theta} = 0$ due to the constraint (39). This is inconsistent with the property that $\bar{\Omega}^* \geq 0$ as $\bar{\tau} \to +\infty$.

Imposing the alternative condition $\bar{N}_2 = \bar{N}_3$ reduces the constraint $\bar{S}_+ = 0$ to $\bar{N}_1(\bar{N}_1 - \bar{N}_2) = 0$. Hence, either $\bar{N}_1 = \bar{N}_2 = \bar{N}_3$ or $\bar{N}_1 = 0$. However, if $\bar{N}_2 = \bar{N}_3 \neq 0$, Eqs. (34) and (35) imply that $\bar{\Sigma}_- = 0$ and $\bar{\Sigma}_+ = 1$, and once more this would correspond to a vacuum model, which is a contradiction.

Hence, the only equilibrium point (with $\bar{H} < 0$) that corresponds to a non-vacuum cosmology with $\gamma = 2$ lies in the Bianchi type I invariant set and is given by

$$\begin{aligned} &d = 1, \quad \bar{H} = -1, \quad \bar{N}_1 = \bar{N}_2 = \bar{N}_3 = 0, \\ &\bar{q} = 2, \quad \bar{\Theta} = 0, \quad \bar{\Sigma}_+^2 + \bar{\Sigma}_-^2 + \bar{\Psi}^2 = 1. \end{aligned} \tag{50}$$

A standard perturbation analysis reveals that the eigenvalues associated with this point are

$$\begin{aligned} &-3, \quad -4, \quad 0, \quad 0, \quad -2 - 4\bar{\Sigma}_+, \quad -4, \\ &-2 + 2\bar{\Sigma}_+ \pm 2\sqrt{3}\bar{\Sigma}_-, \quad -3 - \frac{\sqrt{6}\lambda}{2}\bar{\Psi}. \end{aligned} \tag{51}$$

The two vanishing eigenvalues imply that this is a two-dimensional set of equilibrium points. It follows that the stability of this set of points is sensitive to the kinetic energy of the scalar field. If $\bar{\Psi} < -\sqrt{6}/\lambda$, it represents a saddle, otherwise it may be a source if $\bar{\Sigma}_+ > 1/2$ and $\bar{\Sigma}_-$ is sufficiently small.

As there are no other equilibrium points, this implies in effect that if the scalar field is rolling to large negative values down its potential at a sufficiently fast rate, $\bar{\Psi} < -\sqrt{6}/\lambda$, its equation of state will satisfy $\lim_{\bar{\tau}\to\infty} \gamma > 2$. We may therefore apply the cosmic no hair theorem of the previous Section in this region of parameter space. In other words, the stable attractor solution is the spatially flat and isotropic FRW cosmology corresponding to the equilibrium point $\bar{\Psi} = -\lambda/\sqrt{6}$, $\bar{\Theta} = [(\lambda^2/6) - 1]^{1/2}$ and $\bar{q} = (\lambda^2/2) - 1$. This 'scaling' solution exists only for $\lambda > \sqrt{6}$. In conclusion, therefore, a contracting Bianchi IX cosmology sourced by an orthogonal scalar field rolling down a negative exponential potential with $\bar{\Psi} < -\sqrt{6}/\lambda$ and $\lambda > \sqrt{6}$ collapses into an isotropic singularity, where the sink is the self-similar, spatially flat and isotropic FRW universe with scale factor $a \propto (-t)^{2/\lambda^2}$.

In the next Section we consider such a scalar field model generated from a compactified higher-dimensional cosmology inspired by M*-theory.

4. Collapsing M*-Cosmology

The M*-theory of Hull [5] is a version of M-theory in an eleven-dimensional spacetime of signature (9+2). Compactification of this theory on a timelike circle gives rise to a string theory – termed the type IIA*-theory – with a supergravity limit that is similar to the conventional type IIA string but with a Ramond-Ramond (RR) sector containing form-fields whose kinetic terms have the 'wrong' sign. In this sense, M*-theory may be interpreted as the strongly-coupled limit of the type IIA* theory. Its low-energy limit is a supergravity theory with a bosonic sector given by

$$S_{\mathrm{M}^*} = \int d^{11}x\sqrt{|g|}\left[R + \frac{G_4^2}{48}\right] - \frac{1}{12}\int C_3 \wedge G_4 \wedge G_4, \tag{52}$$

where G_4 represents the field strength of the three–form potential C_3. The sign of the kinetic term for G_4 is determined by supersymmetry.

Leaving aside issues associated with the conceptual problems of extra timelike dimensions (see [5] for a full discussion of such questions within the context of M*-theory), we may consider the Kaluza-Klein compactification of the action (52) on a timelike circle S^1. For the case where only the RR four-form field strength is non-trivial, the ten-dimensional action for the truncated type IIA*-theory is given by

$$S_{\mathrm{IIA}^*} = \int d^{10}x\,\sqrt{-g_s}\left[e^{-\phi_{10}}\left(R_s + (\nabla\phi_{10})^2\right) + \frac{G_4^2}{48}\right], \tag{53}$$

where the ten-dimensional dilaton field, ϕ_{10}, is related to the radius of the eleventh dimension, $e^{r_{11}}$, by $\phi_{10} = 3r_{11}$ and we have performed a conformal transformation:

$$g^{(s)}_{AB} = \Omega^2 g_{AB}, \qquad \Omega^2 \equiv e^{r_{11}} \tag{54}$$

to the string-frame metric, $g^{(s)}_{AB}$.

We will now dimensionally reduce the theory (53) to four dimensions on a six-torus, T^6, where the only dynamical degree of freedom in the internal dimensions is taken to be the breathing mode, β. In other words, we assume that the string-frame metric (54) is given by

$$ds_s^2 = g^{(s)}_{\mu\nu}(x)dx^\mu dx^\nu + e^{2\beta(x)}dy^2, \tag{55}$$

where $dy^2 = \delta_{ij}dy^i dy^j$. We will also Hodge dualize the four-form field strength in ten dimensions to a six-form, F_6, and assume that the only non-zero components of this six-form live on the internal dimensions, i.e., we assume an *ansatz* $F_6 = m\varepsilon_6$, where ε_6 is the volume-form of T^6 and m is a constant. Thus, the effective four–dimensional effective action takes the form

$$S = \int d^4x\sqrt{-g_s}\left[e^{-\phi_4}\left(R_s + (\nabla\phi_4)^2 - 6\,(\nabla\beta)^2\right) - \frac{1}{2}m^2 e^{-6\beta}\right], \tag{56}$$

where $\phi_4 \equiv \phi_{10} - 6\beta$ represents the four-dimensional dilaton field.

Eq. (56) may be expressed in the Einstein-Hilbert form by performing the conformal transformation

$$g^{(e)}_{\mu\nu} = e^{-\phi_4} g^{(s)}_{\mu\nu} \tag{57}$$

and field redefinitions $\tilde{\phi}_4 = \phi_4/\sqrt{2}$ and $\tilde{\beta} = \sqrt{6}\beta$. It follows that the conformally transformed action is given by

$$S = \int d^4x \sqrt{-g_e} \left[R_e - \left(\nabla\tilde{\phi}_4\right)^2 - \left(\nabla\tilde{\beta}\right)^2 + m^2 e^{\sqrt{8}\tilde{\phi}_4 - \sqrt{6}\tilde{\beta}} \right]. \tag{58}$$

Finally, defining a new pair of scalar fields:

$$\begin{aligned} \chi &\equiv -\frac{1}{\sqrt{14}} \left(\sqrt{8}\tilde{\phi}_4 - \sqrt{6}\tilde{\beta}\right) \\ \xi &\equiv \frac{1}{\sqrt{14}} \left(\sqrt{6}\tilde{\phi}_4 + \sqrt{8}\tilde{\beta}\right) \end{aligned} \tag{59}$$

implies that the action (58) is equivalent to

$$S = \int d^4x \sqrt{-g_e} \left[R_e - (\nabla\chi)^2 - (\nabla\xi)^2 + m^2 e^{-\sqrt{14}\chi} \right]. \tag{60}$$

Thus, the potential for the χ-field is negative and sufficiently steep ($\lambda = \sqrt{14}$) for the results of Section III to apply. The scalar degree of freedom ξ will behave as a stiff perfect fluid, but the presence of this field does not affect the conclusions of the previous section, since it is decoupled from the other matter degree of freedom. It therefore becomes dynamically negligible near to the singularity in the case where the collapsing Bianchi type IX cosmology asymptotes to the FRW solution. In this case, the attractor is the power law solution $a_e \propto (-t_e)^{1/7}$, $\chi = (2/\sqrt{14}) \ln(-t_e)$, $\xi = 0$. Transforming back into the four-dimensional string frame via Eq. (57) then implies that $a_s \propto (-t_s)^{-1/5}$, $e^{\phi_4} \propto (-t_s)^{-4/5}$ and $e^{\beta} \propto (-t_s)^{1/5}$. The ten-dimensional string-frame metric (55) is therefore given by

$$ds_s^2 = T_s^{-1/2} dx_3^2 + T_s^{1/2} \left[-dT_s^2 + dy_6^2 \right], \tag{61}$$

where we have defined $T_s \equiv (-t_s)^{4/5}$ and rescaled the coordinates where appropriate.

The scale factors in the eleven-dimensional frame, which we may denote by $\{\hat{a}, e^{\hat{\beta}}, e^{r_{11}}\}$ are related to the corresponding ten-dimensional quantities by Eq. (54) and take the form $\hat{a} = a e^{-r_{11}/2}$ and $\hat{\beta} = \beta - r_{11}/2$, where the eleven-dimensional proper time is given by $\eta \equiv \int dt_s \exp(-r_{11}/2)$. Hence, the eleven-dimensional line element can be expressed as

$$ds_{11}^2 = \hat{\eta}^{-2/3} dx_3^2 + \hat{\eta}^{1/3} \left[-d\hat{\eta}^2 - dt_{11}^2 + dy_6^2 \right], \tag{62}$$

where $\hat{\eta} \equiv (-\eta)^{6/7}$, the coordinate of the eleventh (timelike) dimension is denoted by t_{11} and, for simplicity, we have absorbed any constants of proportionality by a suitable rescaling of the spacetime coordinates.

A background of the form (61) can be interpreted as the cosmological analogue of a domain wall spacetime [17]. More specifically, we may write the metric in the form $ds_{11}^2 = F^{-2/3} dx_3^2 + F^{1/3}[-d\hat{\eta}^2 - dt_{11}^2 + dy_6^2]$, where $\{\hat{\eta}, t_{11}, y\}$ represent the transverse coordinates. Since F is a linear function of $\hat{\eta}$, it may be viewed as a harmonic function on the transverse space, thereby allowing Eq. (62) to be interpreted as a membrane-type solution with a three-dimensional Euclidean world-volume [16, 17]. In particular, it is known that solutions to M*-theory of this form preserve 16 of the 32 supersymmetries

of the theory [17]. It is interesting that such a supersymmetric, higher-dimensional cosmological background is selected from a dynamical point of view when analyzed in terms of an effective four-dimensional, anisotropic cosmology.

5. Conclusion

To summarize, it has been shown that any orthogonal, collapsing Bianchi cosmology with a matter sector comprised of an ultra-stiff fluid with an arbitrary and varying equation of state (subject to the condition that $\gamma > 2$ at all times) approaches spatial isotropy and flatness in the neighbourhood of the big crunch singularity. Of particular interest is the behaviour of the general Bianchi type IX universe. Since the singularity is isotropic, such models do not exhibit chaotic-type (Mixmaster) behaviour near to the big crunch, as is the case when $\gamma < 2$ (see, e.g., [18] and references therein). A related effect has been found in a class of braneworld models with a constant equation of state parameter $\gamma > 1$, where the effective Friedmann equation depends quadratically on the energy density [19]. This can be understood since the fluid in such a model behaves at small spatial volumes as if it was ultra-stiff [20].

The only spatially homogeneous cosmology that we have not considered is the Kantowski-Sachs model, where the G_3 group of isometries acts multiply-transitively. The line element for this cosmology may be written in the form

$$\begin{aligned} & ds^2 = -dt^2 + D_1^2 dx^2 + D_2^2 d\Omega_2^2 \\ & D_1(t) = e^{\beta_0(t) - 2\beta_+(t)}, \qquad D_2(t) = e^{\beta_0(t)+\beta_+(t)} \end{aligned} \tag{63}$$

where $d\Omega_2^2$ is the metric on the two-sphere. Since this spacetime has positive spatial curvature, an extension of our cosmic no hair theorem to include this case would require a set of variables that compactifies the phase space in an analogous way to the variables (26) and (27) for the Bianchi type IX model. However, for $\gamma > 2$, a suitably normalized variable for the energy density that increases monotonically during a collapsing phase has yet to be identified.

Nonetheless, we may gain insight into the asymptotic nature of an ultra-stiff Kantowski-Sachs universe by assuming γ = constant. Following [21], we define $B_{1,2} \equiv D_{1,2}^{-1}$. The Friedmann equation then takes the form

$$3H^2 = \rho + \frac{1}{3}\sigma_+^2 - B_2^2 \tag{64}$$

where $H = \dot{\beta}_0$ and $\sigma_+ = 3\dot{\beta}_+$. It follows that $D \equiv [9H^2 + 3B_2^2]^{1/2}$ is a dominant variable (assuming $\rho \geq 0$), and this implies that compact variables $Q_0 \equiv 3H/D$, $Q_+ \equiv \sigma_+/D$ and $\Omega \equiv 3\rho/D^2 = 1 - Q_+^2$ may be introduced, together with a new time variable $\tau = \int dt D/3$. The curvature variable is $\tilde{K} = B_2^2/(3H^2) = (1 - Q_0^2)/Q_0^2$ from which we deduce that $0 \leq \{Q_0, Q_+, \Omega\} \leq 1$.

It can then be shown [21] that the field equations reduce to the set of evolution equations

$$\frac{dQ_0}{d\tau} = -\left(1 - Q_0^2\right) F \tag{65}$$

$$\frac{dQ_+}{d\tau} = -1 + (Q_0 - Q_+)^2 + Q_0 Q_+ F \tag{66}$$

where

$$F \equiv \frac{3\gamma}{2} - 1 - Q_0 Q_+ + \frac{3}{2}(2-\gamma)Q_+^2 \tag{67}$$

Eq. (65) then implies that all equilibrium points are located at $Q_0^2 = 1$ or $F = 0$. Let us first assume $Q_0^2 \neq 1$ and $F = 0$. It follows from Eqs. (66) and (67) that $Q_+ = \pm(3\gamma - 2)/(4 - 3\gamma)$, but these points are unphysical for $\gamma > 2$. Since we are interested in collapsing universes, we therefore consider the case $Q_0 = -1$. In this case, there are three equilibrium points, where $Q_+ = 0, \pm 1$. A stability analysis indicates that the points $Q_+ = \pm 1$ are saddles and the point $(Q_0, Q_+) = (-1, 0)$ is a sink. This latter point corresponds to the isotropic, spatially flat FRW cosmology. Hence, in the case of a constant, ultra-stiff equation of state, the unique attractor for a collapsing Kantowski-Sachs cosmology is the isotropic and spatially flat universe. This provides strong support for the conjecture that this background represents the attractor for a general ultra-stiff equation of state. It also represents strong motivation for searching for an appropriate set of variables along the lines of Eqs. (26)-(27).

Our no hair results are also of relevance to the nature of spacetime singularities in more general inhomogeneous backgrounds, since according to the Belinskii, Khalatnikov and Lifshitz (BKL) conjecture [22], each spatial point of an inhomogeneous universe evolves effectively as a Bianchi model on the approach to the singularity. In principle, the results derived above can be employed to gain further insight into the nature of inhomogeneous cosmologies [19, 23, 24]. Indeed, similar issues to those considered in this paper have been addressed by Coley and Lim [24], who performed an asymptotic analysis of stiff and ultra-stiff Abelian G_2 and general G_0 spatially inhomogeneous cosmologies, and found that the spatially flat FRW solution is locally stable in the past for $\gamma > 2$. Our results differ in that it was assumed that γ is a constant in [24], whereas we have assumed an arbitrary (differentiable) equation of state $\gamma = \gamma(\rho) > 2$. Furthermore, our no hair result for the spatially homogeneous Bianchi models is a global result, in the sense that the flat FRW solution is the unique attractor at the big crunch.

Finally, the work of [24] is restricted to that of a perfect fluid matter source. We extended our analysis to the case of a collapsing, spatially homogeneous Bianchi type IX model containing a scalar field that self-interacts through a negative potential. In this case, the effective equation of state varies but satisfies $\gamma \geq 2$, with equality corresponding to a vanishing potential. Our cosmic no hair theorem implies that the problem of demonstrating that such models isotropize as they collapse has been reduced to verifying that the potential remains dynamically significant near to the singularity. We have shown that this is possible, for example, in the case of a steep, negative exponential potential. Exponential potentials are of particular interest since they are common in cosmological models inspired by string/M-theory. One such theory that leads naturally to negative potentials is M*-theory, where the potential is generated by non-trivial flux of the form fields [5]. We considered a truncated, dimensionally

reduced effective action for this theory and found that when the isotropic attractor for a collapsing Bianchi type IX cosmology is oxidized back to eleven dimensions, the metric is a supersymmetric solution of the higher-dimensional theory. This is of interest since supersymmetric backgrounds are non-perturbatively exact solutions and therefore provide a valuable framework for investigating the cosmic dynamics in the high curvature regime near to the singularity. We expect that other attractor solutions for more general dimensional reduction schemes will also exhibit similar supersymmetric properties when interpreted in a ten- or eleven-dimensional context.

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